Tests for some Reliability Models with different types of Maintenance

Jean-Yves Dauxois (Univ. of Toulouse, France)

Joint work with Sarah Jomhoori (Univ. Birjand, Iran)

24 of January 2013

Outline

1. Competing Risks Models for different types of maintenance
   - Competing risks formulation
   - Some available probabilistic models
2. Some Tests based on the function Φ
   - Objective
   - Nonparametric Estimation
   - Test of H_0 against H_1
   - Test of H_0 against H_2

Outline

1. Competing Risks Models for different types of maintenance
   - Competing risks formulation
   - Some available probabilistic models

2. Some Tests based on the function Φ
   - Objective
   - Nonparametric Estimation
   - Test of H_0 against H_1
   - Test of H_0 against H_2

Competing risks formulation

Let us consider a repairable system with different types of failure and different types of maintenance.

A competing risks approach can be used to model the observations made on such a system.

$X =$ failure times associated to the failure mode(s) of interest
$\Rightarrow$ Corrective Maintenance (CM)
and
$Y =$ termination time of observation due to other causes, like preventive maintenance or a non-critical failure $\Rightarrow$ Preventive Maintenance (PM)

Observations

Suppose that we observe the r.v. $(T_i, \delta_i)_{i=1,\ldots,n}$ where :

$$\begin{cases} T_i &= X_i \wedge Y_i \\ \delta_i &= 1 + I\{X_i < Y_i\} \end{cases} .$$

The r.v. $(T_i, \delta_i)_{i=1,\ldots,n}$ are supposed to be independent, as the $(X_1, \ldots, X_n)$ and the $(Y_1, \ldots, Y_n)$. But X_i is not necessarily independent from Y_i, for $i = 1, \ldots, n$.

Functions of interest

In general, only the subsurvival functions

$$\begin{aligned} S_2(t) &= P(T > t, \delta = 2) = P(X > t, Y > X), \\ S_1(t) &= P(T > t, \delta = 1) = P(Y > t, X > Y) \end{aligned}$$

are identifiable and not the survival function S_X and S_Y associated to the unobservable lifetimes X and Y.

The conditional survival functions are also estimable and have interesting properties under some repair models, as we will see later:

$$\begin{aligned} CS_2(t) &= P(T > t | \delta = 2) = P(X > t | Y > X), \\ CS_1(t) &= P(T > t | \delta = 1) = P(Y > t | X > Y) \end{aligned}$$

Finally, we will also consider the function which, for all t, gives the probability of "censoring" beyond time t:

$$\Phi(t) = P(\delta = 1 | T > t) = P(Y < X | T > t).$$

The functions $\Phi(\cdot)$, $CS_1(\cdot)$ and $CS_2(\cdot)$ have interesting properties under classical Reliability models of different types of maintenance and can be used to built goodness-of-fit tests.

Outline

1. Competing Risks Models for different types of maintenance
   - Competing risks formulation
   - Some available probabilistic models
2. Some Tests based on the function Φ
   - Objective
   - Nonparametric Estimation
   - Test of H_0 against H_1
   - Test of H_0 against H_2

Independent Competing Risks

The r.v. X and Y are supposed independent.

- This an untestable hypothesis.
- No general results on the behaviour of $\Phi(\cdot)$, $CS_2(\cdot)$ and $CS_1(\cdot)$. All depends on the distribution given to X and Y.
- If X and Y have an exponential distribution, the function $\Phi(\cdot)$ is constant.

Independent Competing Risks with Mixture of Exponentials

Model introduced by Bunea, Cooke and Lindqvist (2003).

The r.v. X and Y are supposed independent with respective survival functions:

$$\begin{aligned} S_X(t) &= p\exp(-\lambda_1 t) + (1-p)\exp(-\lambda_2 t) \\ S_Y(t) &= \exp(-\lambda_y t). \end{aligned}$$

- The function $\Phi(\cdot)$ is strictly increasing when $\lambda_1 \neq \lambda_2$.
- $CS_2(t) \leq CS_1(t)$, for all $t > 0$.

Delay-Time Model

Model introduced by Hokstadt & Jensen (1998).

The r.v. X and Y are supposed to be such that:

$$\begin{aligned} X &= U + V \\ Y &= U + W, \end{aligned}$$

where U, V and W are independent r.v. The r.v. X and Y are thus dependent, but independent given U.

When U, V and W are supposed to be exponentially distributed, we have:

- $\Phi(\cdot)$ is a constant function,
- $CS_1(t) = CS_2(t)$, for all $t > 0$.

Random Signs Censoring Model

Model introduced by Cooke (1993).

The r.v. δ and X are supposed independent, i.e. the sign of $Y - X$ is independent from X.

- The function $\Phi(\cdot)$ has its maximum at the origin,

$$\sup_t \Phi(t) = \Phi(0) = P(\delta = 1).$$

- Cooke (1996) proves that there exists a joint distribution on (X, Y) which satisfies the random signs censoring assumption if, and only if, $CS_2(t) > CS_1(t)$, for all $t > 0$.

(Intensity Proportional) Repair Alert Model

Models introduced by Langseth & Lindqvist (2003) and Lindqvist *et al.* (2006).

The Repair Alert Model is a sub-model of the Random Sign Model where we also have:

$$P(Y \leq y | X = x, Y < X) = \frac{G(y)}{G(x)},$$

where $G(\cdot)$ is an increasing function such that $G(0) = 0$.

The Intensity Proportional Repair Alert Model is obtained with the choice of $G(\cdot) = \Lambda_X(\cdot)$, the cumulative hazard rate function of the time to failure X.

In this case, the function $\Phi(\cdot)$ is decreasing.

Outline

1. Competing Risks Models for different types of maintenance
   - Competing risks formulation
   - Some available probabilistic models
2. Some Tests based on the function Φ
   - Objective
   - Nonparametric Estimation
   - Test of H_0 against H_1
   - Test of H_0 against H_2

Outline

1. Competing Risks Models for different types of maintenance
   - Competing risks formulation
   - Some available probabilistic models

2. Some Tests based on the function Φ
   - Objective
   - Nonparametric Estimation
   - Test of H_0 against H_1
   - Test of H_0 against H_2

Often criteria used to distinguish these models from the data at hand are only graphical, based on estimation of the function $\Phi(\cdot)$, $CS_1(\cdot)$ and $CS_2(\cdot)$.

Some interesting exceptions are:

- Dewan *et al.* (2002) combined the concept of concordance and discordance with U-statistic approach to derive some tests for model selection.
- Langseth and Lindqvist(2006) proposed to use parametric bootstrap to test IPRA model under perfect repair. They have also generalized their result in perfect repair to imperfect repair framework.

Our aim in this paper is to propose a family of tests for testing

$$H_0 : \Phi(t) = \gamma = \Phi(0) = P(\delta = 1), \text{ for all } t > 0,$$

against two alternative hypotheses:

$$H_1 : \Phi(t) < \gamma, \text{ for all } t > 0,$$

or

$$H_2 : \Phi(t) \text{ is a nonconstant decreasing function of } t.$$

In the sequel, we will allow for possibly right censored data, that is we only observe:

$$\left\{ \begin{array}{lcl} T_i^* & = & T_i \wedge C_i \\ \delta_i^* & = & \delta_i I(T_i \leq C_i) \end{array} \right. , \text{ for } i = 1, \ldots, n.$$

The censoring random variables C_i are supposed to be i.i.d. with continuous distribution function H and independent from the other random variables.

Outline

Let us define the counting processes

$$N_j(t) = \sum_{i=1}^{n} I(T_i^* \leq t, \delta_i^* = j), \quad j = 1, 2$$

and the number at risk process

$$Y(t) = \sum_{i=1}^{n} I(T_i^* \geq t).$$

The Kaplan-Meier estimator of $F(t) = P(T \leq t)$ is

$$\widehat{F}(t) = 1 - \prod_{i:t_{(i)}^* \leq t} \left(1 - \frac{\Delta N(t_{(i)}^*)}{Y(t_{(i)}^*)}\right),$$

where $N(\cdot) = \sum_{j=1}^{2} N_j(\cdot)$ and $T_{(1)}^* \leq T_{(2)}^* \leq ... \leq T_{(n)}^*$ are the ordered statistics.

In a competing Risks setup the cumulative incidence function (CIF) are defined as

$$F_j(t) = P(T \leq t, \delta = j), \quad j = 1, 2.$$

Then, the Aalen-Johansen estimators of CIFs, for $j = 1, 2$, are given by

$$\widehat{F}_j(t) = \int_0^t \widehat{S}(u-) \frac{dN_j(u)}{Y(u)},$$

The sub-survival functions is therefore estimated by

$$\widehat{S}_j(t) = \widehat{F}_j(\tau) - \widehat{F}_j(t),$$

where τ is the right endpoint of the support of $F(\cdot)$.

Outline

1. Competing Risks Models for different types of maintenance
   - Competing risks formulation
   - Some available probabilistic models
2. Some Tests based on the function Φ
   - Objective
   - Nonparametric Estimation
   - Test of H_0 against H_1
   - Test of H_0 against H_2

Recall that we want to test:

$$H_0 : \Phi(t) = \gamma = \Phi(0) = P(\delta = 1), \text{ for all } t > 0,$$

against:

$$H_1 : \Phi(t) < \gamma, \text{ for all } t > 0.$$

Note that $H_0 \iff S_1(\cdot)$ and $S(\cdot)$ are proportional
and

$$\begin{aligned} H_1 &\iff \gamma S(t) - S_1(t) > 0, \text{ for all } t > 0 \\ &\iff (1-\gamma)F_1(t) - \gamma F_2(t) > 0 \text{ for all } t > 0 \\ &\iff CS_2(t) > CS_1(t) \text{ for all } t > 0. \end{aligned}$$

Thus, as a criterion of deviation from H_0, one can use:

$$\psi = \psi(w) = \int_0^\tau w(t)\big[(1-\gamma)F_1(t) - \gamma F_2(t)\big]\, dt,$$

where w is a positive weight function. The criterion ψ is null under H_0 and strictly positive under H_1.

Thus a natural test statistic for detecting the alternative H_1 is given by

$$\widehat{\psi} = \int_0^\tau \widehat{w}(t)\big[(1-\widehat{\gamma})\widehat{F}_1(t) - \widehat{\gamma}\widehat{F}_2(t)\big]\, dt,$$

where $\widehat{w}(\cdot)$ is a consistent estimator of $w(\cdot)$ and $\widehat{\gamma} = \widehat{F}_1(\tau) = P\widehat{(\delta = 1)}$.

Theorem

Let us suppose that

$$\int_0^\tau \frac{dF(s)}{\bar{H}(s)} < \infty \tag{1}$$

and as $n \to \infty$

$$\sup_{s\in[0,\tau]} |\widehat{w}(s) - w(s)| \xrightarrow{p} 0.$$

Then, $\sqrt{n}(\widehat{\psi} - \psi)$ *converges weakly to a mean zero normal random variable* $\mathbb{Z}_1$*, with finite variance* σ_1^2*. Under* H_0 *the limiting variance can be expressed in the form of*

$$\sigma_{01}^2 = (1-\gamma)\int_0^\tau \int_0^\tau w(t)w(s)\int_0^{s\wedge t} \frac{dF_1(u)}{\bar{H}(u)} dtds.$$

A short simulation study

Samples under H_1 are simulated from the bivariate distribution:

$$f(x,y) = \frac{1}{2x} e^{-x}, \text{ with } 0 < y < 2x.$$

Variation of the simulation parameters:

- 3 samples sizes: 50, 100 and 200,
- 2 nominal levels: 5% and 2%,
- 4 percentages of censoring: 0%, 10%, 30% and 50%.

Table: Simulation results. Monte Carlo estimates of the power of the test of H_0 against H_1

	Sample Size					
	50		100		200	
Level	5%	2%	5%	2%	5%	2%
Censoring						
0%	0.64	0.42	0.90	0.77	0.99	0.98
10%	0.54	0.34	0.81	0.65	0.98	0.94
30%	0.39	0.21	0.65	0.44	0.91	0.80
50%	0.29	0.15	0.46	0.28	0.74	0.55

Outline

Recall that we want to test:

$$H_0 : \Phi(t) = \gamma = \Phi(0) = P(\delta = 1), \text{ for all } t > 0,$$

against

$$H_2 : \Phi(t) \text{ is a nonconstant decreasing function of } t.$$

Note that:

$$H_1 \iff S_1(x)S_2(y) - S_1(y)S_2(x) \geq 0, \text{ for all } x < y$$

Following Sengupta et al. (1998), one can use

$$\varphi_1(w) = \int\int_{0<x<y<\tau} w(x,y)[S_1(x)S_2(y) - S_1(y)S_2(x)]dxdy,$$

as a measure of deviation from the null hypothesis, where $w(.,.)$ is a suitably chosen positive weight function. Indeed, it is null under H_0 and positive under H_2.

The above double integral may be reduced to products of single integrals by choosing the weight function

$$w(x,y) = K_1(x)K_2(y) - K_1(y)K_2(x),$$

where K_1 and K_2 are positive weight functions with an decreasing ratio. In this case, we get

$$\varphi_1(K_1, K_2) = U_{11}U_{22} - U_{12}U_{21},$$

where

$$U_{ij} = \int_0^\tau K_i(u)S_j(u)du, \quad i,j = 1,2.$$

A consistent estimator of $\varphi_1(K_1, K_2)$ can be used as a test statistic for the problem at hand. Let $\widehat{K}_i, i = 1, 2$, be uniformly consistent estimators of $K_i, i = 1, 2$, respectively. We define the test statistic as

$$\widehat{\varphi}_1 = \widehat{U}_{11}\widehat{U}_{22} - \widehat{U}_{12}\widehat{U}_{21},$$

where

$$\widehat{U}_{ij} = \int_0^\tau \widehat{K}_i(u)\widehat{S}_j(u-)du.$$

Theorem

Let us suppose that as n tends to ∞,

$$\sup_{s\in[0,\tau]} |\widehat{K}_i(s) - K_i(s)| \xrightarrow{p} 0, \quad i = 1, 2.$$

Then, under assumption (1), *one has the weak convergence of* $\sqrt{n}(\widehat{\varphi}_1 - \varphi_1)$ *to a mean zero normal random variable* $\mathbb{Z}_2$, *with finite variance* σ_2^2. *Under the null hypothesis the limiting variance can be expressed in the simplified form of*

$$\sigma_{02}^2 = \frac{1-\gamma}{\gamma^2}\Big\{\int_0^\tau\int_0^\tau b(s)b(t)\Big(\int_0^\tau \frac{dF_1(u)}{\bar{H}(u)} - \int_0^s \frac{dF_1(u)}{\bar{H}(u)} - \int_0^t \frac{dF_1(u)}{\bar{H}(u)} + \int_0^{s\wedge t} \frac{dF_1(u)}{\bar{H}(u)}\Big) dsdt\Big\}$$